

ON CURVES CONTAINED IN CONVEX SUBSETS OF THE PLANE

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ABSTRACT. If $K' \subset K$ are convex bodies of the plane then the perimeter of K' is not greater than the perimeter of K . We obtain the following generalization of this fact. Let K be a convex compact body of the plane with the perimeter p and the diameter d and $r > 1$ be an integer. Let s be the smallest number such that for any curve of length greater than s contained in K there is a straight line intersecting the curve at least in $r + 1$ different points. Then $s = rp/2$ if r is even and $s = (r - 1)p/2 + d$ if r is odd.

Let K be a compact subset of the plane. Recall that the set K is called *convex* if for each pair of points $x, y \in K$ the segment $[x, y]$ connecting these points belongs to K . The set K is called *strictly convex* if for each pair of points $x, y \in K$ the interval $(x, y) = [x, y] \setminus \{x, y\}$ belongs to the interior $\text{int } K$ of the set K . Equivalently, the set K is strictly convex if its boundary dK contains no segments of nonzero length.

Let $I = [0; 1]$ be the unit segment. A *curve* is a continuous map $\varphi : I \rightarrow \mathbb{R}^2$. Let $0 = a_0 < \dots < a_n = 1$ be a sequence. Let $[\varphi]$ be the broken line with the sequence of vertices $\varphi(a_0), \dots, \varphi(a_n)$ that is the union of the segments $\bigcup_{i=0}^{n-1} [\varphi_i; \varphi_{i+1}]$. By $l(a_0, \dots, a_n)$ we denote the length of the broken line $[\varphi]$. We say that the curve φ has *length* $l(\varphi)$ if for each $\varepsilon > 0$ there is $\delta > 0$ such that $|l(\varphi) - l(a_0, \dots, a_n)| < \varepsilon$ for any sequence $0 = a_0 < \dots < a_n = 1$ with $a_{i+1} - a_i < \delta$ for each $0 \leq i \leq n - 1$.

The convex subset K of the plane is called a *convex body* if $\text{int } K \neq \emptyset$. If K is a convex body then the boundary dK of K is an image of a curve and there exists the length $l(dK)$ which is equal to the upper bound of the perimeters of convex polygons¹ inscribed in K [Ale, p. 373], [YB, p.28]. The length of the boundary we shall call the *perimeter* of the body K and shall denote as $p(K)$.

It is well known that $p(K') \leq p(K)$ for every pair of convex bodies K and K' with $K' \subset K$ ([Ale, p. 373], [YB, p.28]). This means that if the image of a curve $\varphi : I \rightarrow K$ is a boundary of a convex body then $l(\varphi) \leq p(K)$. A slightly weaker claim is: if the image of a curve $\varphi : I \rightarrow K$ is a boundary of a strictly convex body then $l(\varphi) \leq p(K)$. The boundary dK' of the strictly convex body K' has the property that every line intersects dK' in at most two points. We are going to generalize the claim showing that $l(\varphi) \leq p(K)$ for every curve $\varphi : I \rightarrow K$ with every line intersecting the image of φ at most in two points. This result is closely related with the following proposition.

Proposition 1. *Let S^1 be a circle and $\varphi : S^1 \rightarrow \mathbb{R}^2$ be a homeomorphic embedding. Then the following conditions are equivalent:*

- (1) $\varphi(S^1)$ is a boundary of a strictly convex body.
- (2) Every line intersects the image of φ at most in two points.
- (3) Every line intersects the image of φ at most in three points.

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¹We can consider a convex polygon as the convex hull of a broken line

Proof. The implication (1) \Rightarrow (2) follows from the definition of the strictly convex body and the implication (2) \Rightarrow (3) is obvious. Now we show the implication (3) \Rightarrow (1). Put $K = \text{conv } \varphi(S^1)$. Since φ is a homeomorphism then K is not a segment and hence K is a convex compact body. Let $x \in S^1$ be an arbitrary point. Let $C \subset \mathbb{R}^2$ be the circle of radius 1 with the center at the point $\varphi(x)$ and $p : \mathbb{R}^2 \setminus \{\varphi(x)\} \rightarrow C$ be the radial projection. Since the map p is continuous the set $p\varphi(S^1 \setminus \{x\})$ is connected and therefore an arc. Suppose that the length of $p\varphi(S^1)$ is greater than π . Then there exist points $x_1, x_2, x_3 \in S^1 \setminus \{x\}$ such that the point $\varphi(x)$ lies in the interior of the triangle with the vertices $p\varphi(x_1), p\varphi(x_2)$ and $p\varphi(x_3)$. Then the point $\varphi(x)$ lies in the interior of the triangle with the vertices $\varphi(x_1), \varphi(x_2)$ and $\varphi(x_3)$. Fix an orientation on the circle S^1 . After a re-enumerating we may suppose that the points lie on the circle S^1 in the order x, x_1, x_2, x_3 . Let $l \subset \mathbb{R}^2$ be a line separating the points $\varphi(x)$ and $\varphi(x_2)$ from the points $\varphi(x_1)$ and $\varphi(x_3)$. Since the images under the map φ of the oriented arcs $(x, x_1), (x_1, x_2), (x_2, x_3)$ and (x_3, x) are linearly connected then each of the images contain a point from the line l . Obtained contradiction shows that the length of the arc $p\varphi(S^1 \setminus \{x\})$ is not greater than π .

Therefore there is a line l going through the point $\varphi(x)$ such that one of the open half planes created by l contains no points of the set $p\varphi(S^1 \setminus \{x\})$ and hence no points of the set $\varphi(S^1)$. This implies that the point $\varphi(x)$ is a boundary point of the set K .

Hence $\varphi(S^1) \subset dK$. Since dK is homeomorphic to a circle [Ale, p. 372] and $\varphi(S^1) \subset dK$ is homeomorphic to a circle the $\varphi(S^1)$ must coincide with dK . The set K is strictly convex since its boundary $\varphi(S^1)$ contains no segments of nonzero length. \square

The main result of the paper is the following

Theorem 1. *Let $K \subset \mathbb{R}^2$ be a convex compact body with perimeter p , diameter d and let $r > 1$ be an integer. Let s be the smallest number such that for any curve $\varphi \subset K$ of length greater than s there is a line intersecting the curve φ at least in $r + 1$ different points. Then*

$$s = \begin{cases} rp/2, & \text{if } r \text{ is even} \\ (r-1)p/2 + d, & \text{if } r \text{ is odd.} \end{cases}$$

Proof. Fix the compact body K and the number r . Let p be the perimeter of K and d be the diameter of K .

The upper bound. Put $s = rp/2$ if r is even and $s = (r-1)p/2 + d$ if r is odd. Let $\varphi : I \rightarrow K$ be a curve of length greater than s . By the definition of the length we can choose numbers $0 = a_0 < \dots < a_n = 1$ such that the length of the broken line with the sequence of vertices $\varphi(a_0), \dots, \varphi(a_n)$ is greater than s . Let l_1, \dots, l_n be the lengths of the segments of the broken line and $\alpha_1, \dots, \alpha_n$ be the angles with these segments and the Ox axis. For an angle $\alpha \in [0; 2\pi]$, by $l(\alpha)$ we denote the sum of the lengths of the projections of the segments l_i onto the line $L_\alpha = \{(x, y) : (x, y) = t(\cos \alpha, \sin \alpha)\}$. Thus $l(\alpha) = \sum l_i |\cos(\alpha - \alpha_i)|$. By $k(\alpha)$ we denote the length of the projection of K onto the line L_α . By Cauchy formula [Had, 6.1.5]

$$\int_0^{2\pi} k(\alpha) d\alpha = 2p.$$

At first suppose that r is even. To the rest of the proof it suffice to show that there is an angle α' such that $l(\alpha') > rk(\alpha')$. Suppose the contrary. Then

$$\begin{aligned} 2rp &= r \int_0^{2\pi} k(\alpha) d\alpha \geq \int_0^{2\pi} l(\alpha) d\alpha = \int_0^{2\pi} \sum l_i |\cos(\alpha - \alpha_i)| d\alpha = \\ &= \sum \int_0^{2\pi} l_i |\cos(\alpha - \alpha_i)| d\alpha = \sum \int_0^{2\pi} l_i |\cos(\beta)| d\beta = \sum 4l_i > 4s = 2rp, \end{aligned}$$

the contradiction.

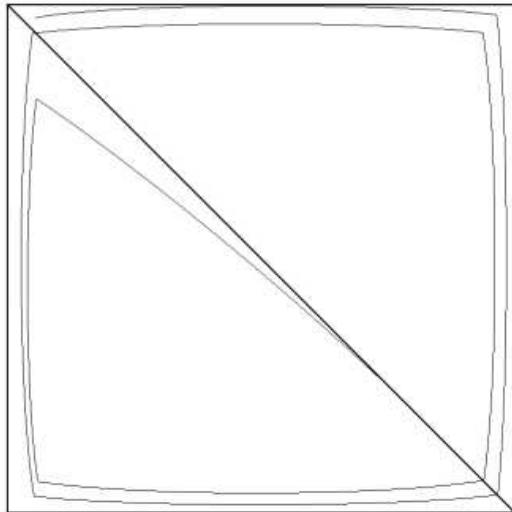
Now suppose that r is odd. Add to the broken line the segment connecting its ends. Let the length of the segment be l_0 and α_0 be the angle with the segment and the Ox axis. To the rest of the proof it suffice to show that there is an angle α' such that $l(\alpha') > rl_0 |\cos(\alpha' - \alpha_0)| + (r-1)(k(\alpha') - l_0 |\cos(\alpha' - \alpha_0)|) = (r-1)k(\alpha') + l_0 |\cos(\alpha' - \alpha_0)|$. That is because each line not intersecting l_0 and the vertices of the broken line intersects the broken line an even number of times.

Suppose the contrary. Then

$$\begin{aligned} 2(r-1)p + 4l_0 &= \int_0^{2\pi} (r-1)k(\alpha) + l_0 |\cos(\alpha - \alpha_0)| d\alpha \geq \int_0^{2\pi} l(\alpha) d\alpha = \int_0^{2\pi} \sum l_i |\cos(\alpha - \alpha_i)| d\alpha = \\ &= \sum \int_0^{2\pi} l_i |\cos(\alpha - \alpha_i)| d\alpha = \sum \int_0^{2\pi} l_i |\cos(\beta)| d\beta = \sum 4l_i > 4s = 2(r-1)p + 4d, \end{aligned}$$

which yields the contradiction since $l_0 \leq d$. \square

The lower bound. The idea of the proof is the following. Put $n = \lfloor r/2 \rfloor$. If r is even, let the curve go n times around the perimeter, curving slightly to avoid any straight lines but remaining convex. If r is odd, let the curve go almost n times around the perimeter, then down the diameter, again always curving slightly. The reason we go “almost” n times around the perimeter is to ensure that any straight line intersecting this “curved diameter” twice, will intersect the perimeter-curve at most $r-2$ times rather than $r-1$. The picture illustrates the construction when K is a square and $r = 5$.



Now more precisely. Let $\varepsilon > 0$. At first we suppose that r is even. There is a curve $\varphi_1 : I \rightarrow K$ such that $\varphi_1(0) = \varphi_1(1)$, $|l(\varphi_1) - p| < \varepsilon$ and $\varphi_1(I)$ is the boundary of a

strictly convex body. Choose a number $\delta_1 > 0$ such that $l(\varphi_1([1 - \delta_1; 1])) < \varepsilon$. Similarly, there is a curve $\varphi_2 : I \rightarrow K$ such that $\varphi_2(0) = \varphi_2(1)$, $|l(\varphi_1) - l(\varphi_2)| < \varepsilon$, $\varphi_2(I)$ is the boundary of a strictly convex body, $\varphi_2(0) = \varphi_1(1 - \delta)$ and $\varphi_2((0; 1)) \subset \text{int conv } \varphi_1(I)$. Choose a number $\delta_2 > 0$ such that $l(\varphi_2([1 - \delta_2; 1])) < \varepsilon$. Similarly to the previous we can construct the curve $\varphi_3 : I \rightarrow K$ and so on. From the curves $\varphi_1, \dots, \varphi_n$ it is easily to construct the curve φ intersecting every straight line in at most r points.

Now suppose that r is odd. To construct the curve φ we shall proceed similarly to the case of even n . By induction we can construct the curves $\varphi_1, \dots, \varphi_n$ such that

- (i) $|\text{diam}(\varphi_1(I)) - \text{diam}(K)| < \varepsilon$,
- (ii) $|\text{diam}(\varphi_i(I)) - \text{diam}(\varphi_{i-1}(I))| < \varepsilon$ for each i ,
- (iii) there is a number α_i such that $|\text{diam}(\varphi_i(I)) - |\varphi_i(0) - \varphi_i(\alpha_i)|| < \varepsilon$ for each i .

Now choose a number $\delta_n > 0$ such that $|\text{diam}(\varphi_n(I)) - |\varphi_n(1 - \delta_n) - \varphi_n(\alpha_n)|| < \varepsilon$ and $l(\varphi_n([1 - \delta_n; 1])) < \varepsilon$. There exists a curve $\varphi_{n+1} : I \rightarrow K$ such that

- (i) $\varphi_{n+1}(0) = \varphi_n(1 - \delta_n)$ and $\varphi_{n+1}(1) = \varphi_n(\alpha_n)$,
- (ii) $\varphi_{n+1}((0; 1))$ lies in the interior of the triangle with the vertices $\varphi_n(0)$, $\varphi_n(\alpha_n)$ and $\varphi_n(1 - \delta_n)$,
- (iii) $\varphi_{n+1}(I) \cup [\varphi_n(1 - \delta_n); \varphi_n(\alpha_n)]$ is the boundary of a convex body,
- (iv) no three points of the set $\varphi_{n+1}(I)$ lie on a straight line.

From the curves $\varphi_1, \dots, \varphi_{n+1}$ it is easy to construct the curve φ intersecting every straight line in at most r points. \square

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